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On slowly-varying Stokes waves

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A WKB-perturbation technique is applied to study the slow modulation of a Stokes wave train on the surface of water. It is found that new terms directly representing modulation rates must be included to extend the scope of validity of Whitham's theory based on an averaged Lagrangian. Two examples are discussed. In the first, a monochromatic wave normally incident on a mild beach is studied and the local rate of depth variation is found to affect the wave phase. In the second, the 'side-band instability' problem of Benjamin & Feir is discussed from both linear and non-linear points of view.

1. Introduction

The slow evolution of non-linear dispersive waves has received much attention since the work of Longuet-Higgins & Stewart (1962) and Whitham (1962) on water waves. For nearly periodic wave trains Whitham (1965*a*, *b*) generalized the method of averaging and deduced the basic equations governing the slow modulation of the amplitude, the wave-number, etc., with respect to both space and time. He further introduced the alternative of assuming an averaged Lagrangian and showed that these equations followed from a variational principle (1967*a*, *b*). While Whitham's general approach has since found wide applications, its justification by formal perturbation schemes has also been given by Luke (1966) for a non-linear Klein–Gordon equation and by Hoogstraten (1968, 1969) for both deep and shallow water waves. In Hoogstraten's work two ordering parameters characterizing the wave steepness and the modulation rate were distinguished. In the present paper, we shall show that by allowing these two to be the same ϵ, the scope of Whitham theory is extended. Of special significance is the fact that terms of higher derivatives and dispersive types are added to Whitham's modulation equations.

These new terms are second-order corrections due to slow modulation rates and not due to non-linearity, and some of them may be foreseen from heuristically examining the linear terms alone. To illustrate we consider the following potential

$$\Phi(X,T) = \epsilon\frac{ga}{\omega}e^{kZ}\cos(kX-\omega T), \tag{1.1}$$

where a, k, ω are slowly varying functions of X and T with the characteristic scales $O(1/\epsilon)$ longer than the primary wavelength and period. Now, from the free-surface boundary condition (cf. (2.5)), it is obvious that the term ϕ_{TT} will bring out a term like $(a/\omega)_{TT}$ of the order $O(\epsilon^2)$, i.e. same as the leading non-linear terms

$O(\epsilon ka)^2$, to the dispersion relation (cf. (7.1)). It is this type of term which modifies Whitham's theory in a significant way.

Two special cases are treated as illustrations. In the first, we study a quasi-steady wave train incident normally on a beach. Earlier results of induced mass flux, mean sea level change, etc., are obtained in a purely formal manner. Besides, there is now a direct influence of bottom slope upon the wavephase. The phase change depends on the vertical co-ordinate, hence the normal to the surface of constant phase is no longer horizontal.

One of the interesting applications of Whitham's theory is on the side-band instability of two-dimensional Stokes wave originated by Benjamin & Feir (1967) and Benjamin (1967). This problem has been generalized by Benney & Newell (1967) and Benney & Roskes (1970). With quite different mathematical techniques the results by these authors agree with each other on most but not all essential points. In particular, Whitham's theory is restricted to near-zero side-band width, while others are not. In the present paper it will be shown that with the additional dispersive terms this restriction is finally removed and the fuller Whitham equations are capable of yielding all the results by other methods. Lastly, the resemblance between Boussinesq equations of shallow water waves and the slow modulation equations of deep water waves is pointed out, and its non-linear implications are further discussed in the light of recent numerical studies of the former by Madsen & Mei (1969*a*, *b*).

2. Formulation and perturbation expansions

We use X, Y, Z and T to denote natural space and time variables:

$$\mathbf{X} = (X, Y) = \text{the horizontal co-ordinates,}$$

Z = vertical co-ordinate, positive upward. We also distinguish the three-dimensional gradient operator

$$\nabla_3 = \left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right)$$

from the horizontal gradient operator

$$\nabla_2 = \left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, 0\right).$$

The basic governing equations for water waves are

$$\nabla_3^2\Phi = \nabla_2^2\Phi + \Phi_{ZZ} = 0, \quad -h(\mathbf{X}) \leqslant Z \leqslant \eta(\mathbf{X}, T), \tag{2.1}$$

$$\Phi_{TT} + g\Phi_Z + \left(\frac{\partial}{\partial T} + \tfrac{1}{2}\mathbf{u}\,.\,\nabla_3\right)|\mathbf{u}|^2 = 0, \quad Z = \eta(\mathbf{X}, T), \tag{2.2}$$

$$\Phi_Z + \nabla_2\Phi\,.\,\nabla_2 h = 0, \quad Z = -h(\mathbf{X}), \tag{2.3}$$

where $\mathbf{u} = \nabla_3\Phi$. The free surface elevation is related to the potential by

$$\eta = -\frac{1}{g}(\Phi_T + \tfrac{1}{2}|\mathbf{u}|^2), \quad Z = \eta(\mathbf{X}, T). \tag{2.4}$$

For small amplitude waves we may expand the free surface conditions (2.2) and (2.4) about $Z = 0$, yielding,

$$\sum_{\nu=0}^{\infty} \frac{\eta^\nu}{\nu!} \frac{\partial^\nu}{\partial Z^\nu} \left[\Phi_{TT} + g\Phi_Z + \left(\frac{\partial}{\partial T} + \tfrac{1}{2}\mathbf{u}.\nabla_3 \right) |\mathbf{u}|^2 \right] = 0 \quad (Z = 0), \tag{2·5}$$

$$g\eta + \sum_{\nu=0}^{\infty} \frac{\eta^\nu}{\nu!} \frac{\partial^\nu}{\partial Z^\nu} [\Phi_T + \tfrac{1}{2}|\mathbf{u}|^2] = 0 \quad (Z = 0). \tag{2·6}$$

In addition to the primary scales of wavelength and period we anticipate much longer scales of slow modulation. Let ϵ be the small ordering parameter characterizing both the wave steepness and the rate of modulation. In particular the depth is assumed to change slowly so that $h = h(\epsilon\mathbf{X})$. We introduce the following slow variables x, y, and t by

$$\mathbf{x} = (x, y) = \epsilon\mathbf{X} = \epsilon(X, Y), \quad t = \epsilon T, \quad z = Z. \tag{2.7}$$

Now the following expansions of WKB type are assumed:†

$$\left.\begin{aligned} \Phi(\mathbf{X}, Z, T) &= \sum_{n=1}^{\infty} \epsilon^n \sum_{m=-n}^{+n} \phi^{(n,m)}(\mathbf{x}, z, t)\, e^{im\psi/\epsilon}, \\ \eta(\mathbf{X}, T) &= \sum_{n=1}^{\infty} \epsilon^n \sum_{m=-n}^{+n} \eta^{(n,m)}(\mathbf{x}, t)\, e^{im\psi/\epsilon}. \end{aligned}\right\} \tag{2.8a, b}$$

For real Φ and η we require $\phi^{(n,m)}$ and $\eta^{(n,m)}$ to be the complex conjugates of $\phi^{(n,-m)}$ and $\eta^{(n,-m)}$. The phase function $\psi(\mathbf{x}, t)$ then defines the wave-number $\mathbf{k}$ and frequency ω in the usual way:

$$\left.\begin{aligned} \mathbf{k}(\mathbf{x}, t) &\equiv (1/\epsilon)\nabla_2\psi \equiv \nabla\psi, \\ \omega(\mathbf{x}, t) &\equiv -(1/\epsilon)\psi_T \equiv -\psi_t, \end{aligned}\right\} \tag{2.9a, b}$$

where $\nabla = (\partial/\partial x, \partial/\partial y, 0)$. Equations (2.9) also imply that

$$\nabla \times \mathbf{k} = 0 \quad \text{and} \quad \mathbf{k}_t + \nabla\omega = 0. \tag{2.10a, b}$$

In addition $\psi, \mathbf{k}, \omega$ are also expanded

$$\psi = \sum_{j=0}^{\infty} \epsilon^{2j}\psi_{2j}, \quad \mathbf{k} = \sum_{j=0}^{\infty} \epsilon^{2j}\mathbf{k}_{2j}, \quad \omega = \sum_{j=0}^{\infty} \epsilon^{2j}\omega_{2j}. \tag{2.11a, b, c}$$

Upon substituting (2.8) and (2.11) into (2.1), (2.3), (2.5) and (2.6) and separating different orders and harmonics, we obtain a set of boundary-value problems involving ordinary differential equations. For each pair of indices (n, m) we have

$$\left.\begin{aligned} &\phi_{zz}^{(n,m)} - m^2k_0^2\phi^{(n,m)} = R^{(n,m)}(\mathbf{x}, z, t) \quad (-h \leqslant z \leqslant 0), \\ &g\phi_z^{(n,m)} - m^2\omega_0^2\phi^{(n,m)} = G^{(n,m)}(\mathbf{x}, t) \quad (z = 0), \\ &\phi_z^{(n,m)} = F^{(n,m)}(\mathbf{x}, t) \quad (z = -h), \\ \text{and} \qquad &\eta^{(n,m)} = \frac{1}{g}[im\omega_0\phi^{(n,m)}(\mathbf{x}, 0, t) - H^{(n,m)}(\mathbf{x}, t)] \quad (z = 0), \end{aligned}\right\} \tag{2.12a, b, c, d}$$

† Early ideas of the present method may be found in Benney & Rosenblat (1964), Bretherton (1968) and Mei, Tlapa & Eagleson (1968).

where $R^{(n,m)}$, $G^{(n,m)}$, $F^{(n,m)}$ and $H^{(n,m)}$ depend on terms of orders lower than n. Their explicit form will be given when needed. Thus a kind of separation of variables is formally achieved; the task remains to solve the system in succession.

3. General procedure of solution

Because of the increasing complexities at higher orders, it is helpful to study some general features before going into details. We shall first present the formal solution to (2.12). In particular the zeroth harmonic is simplest. By taking $m = 0$, we obtain a first-order equation for $\phi_z^{(n,0)}$ with two boundary conditions. The solution satisfying (2.12a) and (2.12c) may be given as

$$\phi_z^{(n,0)} = \int_{-h}^{z} R^{(n,0)}\,dz + F^{(n,0)}, \tag{3.1}$$

but this gives no further information on $\phi^{(n,0)}$. The extra condition (2.12b) on the free-surface gives a solvability condition relating $R^{(n,0)}$, $F^{(n,0)}$ and $G^{(n,0)}$,

$$\int_{-h}^{0} R^{(n,0)}\,dz + F^{(n,0)} = \frac{1}{g} G^{(n,0)}. \tag{3.2}$$

Now it can be shown that

$$R^{(n,0)} = -\nabla^2 \phi^{(n-2,0)} \quad \text{and} \quad F^{(n,0)} = -\nabla h \,.\, [\nabla \phi^{(n-2,0)}]_{z=-h}. \tag{3.3}$$

It follows by using the Leibniz rules that (3.2) is equivalent to

$$\nabla \,.\, \int_{-h}^{0} \nabla \phi^{(n-2,0)}\,dz = -\frac{1}{g} G^{(n,0)}, \tag{3.4}$$

and (3.1) gives

$$\phi_z^{(n,0)} = -\nabla \,.\, \int_{-h}^{z} \nabla \phi^{(n-2,0)}\,dz. \tag{3.5}$$

Equation (3.4), which is the consequence of the nth order, gives a relation for $\nabla \phi^{(n-2,0)}$; in other words, restriction on $\nabla\phi^{(n-2,0)}$ is found at two orders later. We may note, physically, that $\epsilon^n \phi_z^{(n,0)}$ contributes at the nth order to the vertical mean current while $\epsilon^n \nabla_2 \phi^{(n,0)} = \epsilon^{n+1} \nabla \phi^{(n,0)}$ contributes at the $(n+1)$th order to the horizontal mean current.

For other harmonics $m \neq 0$, the solution that satisfies (2.12a) and (2.12c) is formally,

$$\phi^{(n,m)} = A^{(n,m)} \cosh mQ + \frac{1}{mk_0} F^{(n,m)} \sinh mQ + \frac{1}{mk_0^2}\left\{\sinh mQ \int_0^Q R^{(n,m)} \cosh mQ'\,dQ' - \cosh mQ \int_0^Q R^{(n,m)} \sinh mQ'\,dQ'\right\}, \tag{3.6}$$

where $Q = k_0(z+h)$, $k_0 = |\mathbf{k}_0|$. Substituting (3.6) into the free surface condition (2.12b) we have

$$(k_0 \sinh mq - mk_\infty \cosh mq)\left\{mA^{(n,m)} - \frac{1}{k_0^2}\int_0^q R^{(n,m)} \sinh mQ\,dQ\right\} + (k_0 \cosh mq - mk_\infty \sinh mq)\left\{\frac{1}{k_0} F^{(n,m)} + \frac{1}{k_0^2}\int_0^q R^{(n,m)} \cosh mQ\,dQ\right\} = \frac{G^{(n,m)}}{g}, \tag{3.7}$$

where $k_\infty \equiv \omega_0^2/g, q \equiv k_0 h$. In general, for all $n, m = 2, 3, 4 \ldots, n$, (3.7) fixes the coefficient $A^{(n,m)}$ uniquely. When $m = 1$ the coefficient $A^{(n,m)}$ drops out explicitly from (3.7), which takes on special significance, as discussed below.

For $n = 1, m = 1$, it is easy to show that $R^{(1,1)} = F^{(1,1)} = G^{(1,1)} = H^{(1,1)} = 0$, the system (2.12$a, b, c$) is homogeneous, leading to the solution

$$\phi^{(1,1)} = -(i/\omega_0)\, gA^{(1,1)} \cosh Q, \quad \eta^{(1,1)} = A^{(1,1)} \cosh q = \tfrac{1}{2}a, \tag{3.8}$$

whereas, from (3.7), the classical dispersion relation is obtained.

$$k_0 \tanh q = k_\infty. \tag{3.9}$$

For $m = 1, n = 2, 3, \ldots$, (3.7) corresponds to the solvability condition for the inhomogeneous boundary-value problem defined by (2.12); its specific form is

$$F^{(n,1)} + \frac{1}{k_0}\int_0^q R^{(n,1)} \cosh Q\, dQ = \frac{\cosh q}{g} G^{(n,1)}. \tag{3.10}$$

Now $F^{(n,1)}$ and $R^{(n,1)}$ depend on the terms $A^{(n-1,1)}$, ω_{n-1} and $\mathbf{k}_{n-1}$ which are yet to be determined. For example, at the stage $n = 2$, (3.10) provides a condition for $A^{(1,1)}$; at $n = 3$ it provides a relation between $\mathbf{k}_2$ and ω_2, etc., ...

4. Equations governing the slow modulation

The formal solutions to (2.12a, b, c) are straightforward and the results and some important information regarding them are summarized in appendix A. Substituting (A 4a, b, c) into the solvability condition (3.10),

$$-\nabla h \,.\, \{i\mathbf{k}_0 \phi^{(1,1)}\}_{z=-h} - \frac{1}{k_0}\int_0^q \{i\nabla \,.\, [\mathbf{k}_0 \phi^{(1,1)}] + i\mathbf{k}_0 \,.\, \nabla\phi^{(1,1)}\} \cosh Q\, dQ$$
$$= \frac{1}{g}\cosh q \{i[\omega_0 \phi^{(1,1)}]_t + i\omega_0 \phi_t^{(1,1)}\}_{z=0}. \tag{4.1}$$

Multiplying both sides by $A^{(1,1)}$ and making use of the Leibniz rule we obtain

$$\nabla \,.\, \int_{-h}^0 \mathbf{k}_0 [\phi^{(1,1)}]^2\, dz = -\frac{1}{g}\frac{\partial}{\partial t}\{\omega_0 [\phi^{(1,1)}]^2\}_{z=0}, \tag{4.2}$$

which can be integrated to give the equation of energy conservation,

$$\frac{\partial}{\partial t}\left(\frac{E}{\omega_0}\right) + \nabla \,.\, \left(\mathbf{C}_g \frac{E}{\omega_0}\right) = 0, \tag{4.3a}$$

where
$$\mathbf{C}_g = \frac{\mathbf{k}_0}{2}\frac{\omega_0}{k_0^2}\left(1 + \frac{2q}{\sinh 2q}\right), \quad E = \tfrac{1}{2} g a^2. \tag{4.3b, c}$$

It can be shown that

$$G^{(3,0)} = \frac{\partial}{\partial t}(g\eta^{(2,0)}) - 2\nabla \,.\, \{\mathbf{k}_0 \omega_0 [\phi^{(1,1)}]^2\}_{z=0}. \tag{4.4}$$

Substituting this into the solvability condition (3.4), we obtain a conservation equation for the second-order mean quantities $\eta^{(2,0)}$ and $\nabla\phi^{(1,0)}$ as follows:

$$\frac{\partial \eta^{(2,0)}}{\partial t} + \nabla \,.\, \left(h\nabla\phi^{(1,0)} + \frac{\mathbf{k}_0}{\omega_0} E\right) = 0. \tag{4.5}$$

Another equation relating the mean quantities may be obtained by differentiating equation (A 10), appendix A, with respect to $\mathbf{x}$,

$$\frac{\partial \nabla \phi^{(1,0)}}{\partial t} + \nabla\{g\eta^{(2,0)} + \tfrac{1}{2}k_\infty(\sigma^2-1)E\} = 0. \tag{4.6}$$

For $n = 3$, $m = 1$, we may obtain

$$\left.\begin{aligned} R^{(3,1)} &= -\nabla^2\phi^{(1,1)} - i\nabla.(\mathbf{k}_0\phi^{(2,1)}) - i\mathbf{k}_0.\nabla\phi^{(2,1)} + 2\mathbf{k}_0.\mathbf{k}_2\phi^{(1,1)},\\ F^{(3,1)} &= -\nabla h.(\nabla\phi^{(1,1)} + i\mathbf{k}_0\phi^{(2,1)})_{z=-h},\\ G^{(3,1)} &= ia\omega_0 g\left(\xi_S + \xi_D + \xi_U + \xi_T - \frac{\omega_2}{\omega_0}\right)\end{aligned}\right\} \tag{4.7a, b, c}$$

where

$$\left.\begin{aligned} \xi_S &= \frac{k_0^2a^2}{16}(9\sigma^4 - 10\sigma^2 + 9), \quad \sigma \equiv \coth k_0 h,\\ \xi_D &= \tfrac{1}{2}k_\infty(\sigma^2-1)\eta^{(2,0)},\\ \xi_U &= \frac{\mathbf{k}_0}{\omega_0}.\nabla\phi^{(1,0)},\\ \xi_T &= (\phi_{tt}^{(1,1)}/2\omega_0^2\phi^{(1,1)})_{z=0}.\end{aligned}\right\} \tag{4.8a, b, c, d}$$

Substituting (4.7) into the solvability condition (3.10), and integrating, we obtain a relation between $\mathbf{k}_2$ and ω_2:

$$\omega_2 = \omega_0[\xi_S + \xi_D + \xi_U + \xi_T + \xi_X] + \mathbf{k}_2.\mathbf{C}_g \tag{4.9}$$

where

$$\xi_X = \frac{1}{a\omega_0\cosh q}\left\{\nabla h.[-i\nabla\phi^{(1,1)} + \mathbf{k}_0\phi^{(2,1)}]_{z=-h} + \frac{1}{k_0}\int_0^q [-i\nabla^2\phi^{(1,1)} + \nabla.(\mathbf{k}_0\phi^{(2,1)}) + \mathbf{k}_0.\nabla\phi^{(2,1)}]\cosh Q\,dQ\right\}. \tag{4.10}$$

Combining this with (2.10*b*) gives the following conservation equation for wavenumber accurate to the second order $O(\epsilon^2)$:

$$\frac{\partial \mathbf{k}}{\partial t} + \nabla\{\omega_0[1+\epsilon^2\xi] + \epsilon^2\mathbf{k}_2.\mathbf{C}_g\} = O(\epsilon^4), \tag{4.11}$$

where

$$\xi = \xi_S + \xi_D + \xi_U + \xi_X + \xi_T. \tag{4.12}$$

Physically the terms $\xi_S, \xi_D, \xi_U, \xi_X, \xi_T$ represent respectively the effects of Stokes amplitude dispersion, mean depth change, mean current, spatial modulation and temporal modulation.

Now, equations (4.3), (4.5) and (4.6) are completely equivalent to three of Whitham's equations (1967*a*, equations (39), (40), (42)) derived by averaging techniques. Note that all three are of order $O(\epsilon^2)$ on the whole, hence they can be written by changing all $\mathbf{k}_0$ to $\mathbf{k}$ with an error of $O(\epsilon^4)$ only. To compare (4.11) with the remaining equation of Whitham (1967*a*, equation (41)), we use the fact that

$$\omega = \omega_0 + \epsilon^2\omega_2 + \ldots \quad \text{and} \quad \mathbf{k} = \mathbf{k}_0 + \epsilon^2\mathbf{k}_2 + \ldots,$$

hence

$$k_0 = k[1 - \epsilon^2\mathbf{k}.\mathbf{k}_2/k^2 + O(\epsilon^4)]. \tag{4.13}$$

It follows that

$$\omega_0(k_0) = \omega_0(k) - \epsilon^2\mathbf{k}_2.\mathbf{C}_g + O(\epsilon^4), \tag{4.14}$$

where

$$\omega_0^2(k) = gk\tanh kh. \tag{4.15}$$

Thus, $$\frac{\partial \mathbf{k}}{\partial t}+\nabla\{\omega_0[1+\epsilon^2\xi]\} = O(\epsilon^4). \tag{4.16}$$

Referring to (4.12), we point out that (4.16) differs from Whitham's corresponding equation by two terms ξ_X and ξ_T, all others being identical. As can be seen from their definitions, ξ_X and ξ_T both involve terms twice differentiated with respect to space and time respectively. Thus only when the modulation rate is very much less than the wave steepness (i.e. modulation is characterized by scales much longer than $O(1/\epsilon)$) can ξ_X and ξ_T be ignored, in which case (4.16) reduces to Whitham's exactly.

Now it can also be seen that these new terms, arising from $\phi^{(1,1)}$ and $\phi^{(2,1)}$ only depend on the amplitude a in a linear way; therefore, they must arise from linear terms in the governing equations. To demonstrate their effects we briefly treat two examples in §5 and §6.

5. Monochromatic waves over variable bottom

This example singles out the effect of spatial modulation rate for the frequency is kept as a pure constant and all a, $\mathbf{k}$, etc., *now real*, are independent of t. The following classical results are immediate from (4.3), (4.5) and (4.6):

$$\nabla.\left(\mathrm{C}_g\frac{E}{\omega_0}\right) = 0. \tag{5.1}$$

$$h\nabla\phi^{(1,0)} = -\frac{\mathbf{k}_0}{\omega_0}E+\text{constant}, \tag{5.2}$$

$$\eta^{(2,0)} = -\frac{1}{g}\frac{k_\infty}{2}(\sigma^2-1)E. \tag{5.3}$$

Equation (5.1) represents well-known energy conservation in refracting waves which can be further integrated explicitly in two dimensions:

$$\frac{a}{a_\infty} = \left[\frac{k}{k_\infty}\left(\frac{\sinh 2q}{1+\sinh 2q}\right)\right]^{\frac{1}{2}}, \tag{5.4}$$

where a_∞ is the first-order amplitude in infinitely deep water. Equation (5.2) states the constancy of mass flux due to waves, first obtained by Whitham (1962). In the case of a closed beach, the constant vanishes, and $\nabla\phi^{(1,0)}$ corresponds to the return current. Equation (5.3) gives the mean sea level change first predicted by Longuet-Higgins & Stewart (1962) as a consequence of radiation stresses. In the rest of §5 we restrict ourselves to the two-dimensional case, hence $\mathbf{k} = (k, 0)$ and $h = h(x)$, etc. With $\omega_2 = 0$ and $\xi_T = 0$, it follows from (4.9) that

$$k_2 = -\frac{\omega_0}{Cg}(\xi_S+\xi_D+\xi_U+\xi_X). \tag{5.5}$$

The expression for ξ_X is lengthy and is given in appendix B. Numerical results have been obtained for a plane closed beach with $h = -x$ and normal incidence.† The notable feature is that ξ_D, ξ_U and ξ_X are positive and tend to shorten the

† The region shoreward of the breaking zone is excluded.

waves, while ξ_S is negative and tends to lengthen the waves. With particular combination of parameters the whole correction k_2 may be zero; it has in fact been observed in experiments that sometimes the linearized dispersion relation gives better prediction than an incomplete non-linear one using only the Stokes term (Eagleson 1956). Numerical results are shown in figure 1.

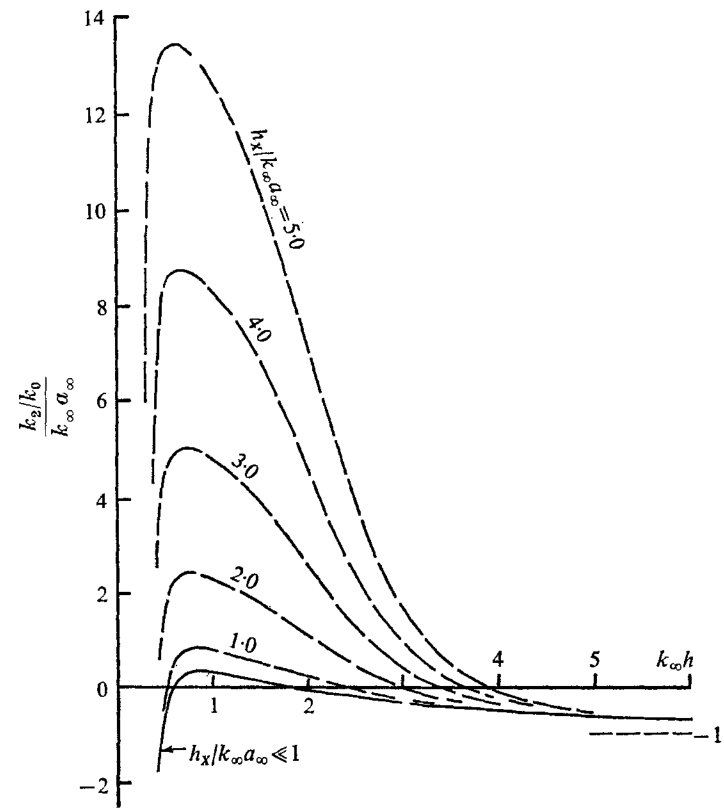

FIGURE 1. Variation of total second-order modification of wave-number with respect to still water depth. Normal incidence on a plane beach. Solid curve represents the case where ξ_X is negligible.

Another feature of interest is that $\phi^{(2,1)}$ is out of phase with $\phi^{(1,1)}$, etc., and $\eta^{(2,1)} = 0$ (cf. (A 8)). Now by rewriting

$$[\epsilon\phi^{(1,1)} + \epsilon^2\phi^{(2,1)}]\, e^{i\psi/\epsilon} = \epsilon\phi^{(1,1)}\, e^{i(\psi+\epsilon^2\delta)/\epsilon} + O(\epsilon^3), \tag{5.6}$$

where $$\delta(\mathbf{x}, z) = -[\alpha_1(Q-q) + \alpha_2(Q\tanh Q - q\tanh q) + \alpha_3(Q^2-q^2)] \tag{5.7}$$

correct to the order $O(\epsilon^2)$, we have

$$\Phi = \epsilon\phi^{(1,0)} + 2i[\epsilon\phi^{(1,1)}\sin(1/\epsilon)(\psi+\epsilon^2\delta) + \epsilon^2\phi^{(2,2)}\sin(2/\epsilon)(\psi+\epsilon^2\delta)] + O(\epsilon^3). \tag{5.8}$$

Due to the z dependence of δ a surface of constant phase is no longer vertical, but is given by

$$\frac{1}{\epsilon}\int^x k\,.\,dx + \epsilon\delta(x,z) - t_0 = \text{constant} \tag{5.9}$$

for fixed $t = t_0$. It can be shown that this surface is perpendicular to both $z = 0$ and the bottom and is approximately given by the following,

$$x - x_0 = \frac{\delta(x_0, z)}{k_0(x_0)} + O(\epsilon). \tag{5.10}$$

Typical geometry of the equal phase curves is shown in figure 2, indicating concavity toward the shore (Battjes 1968).

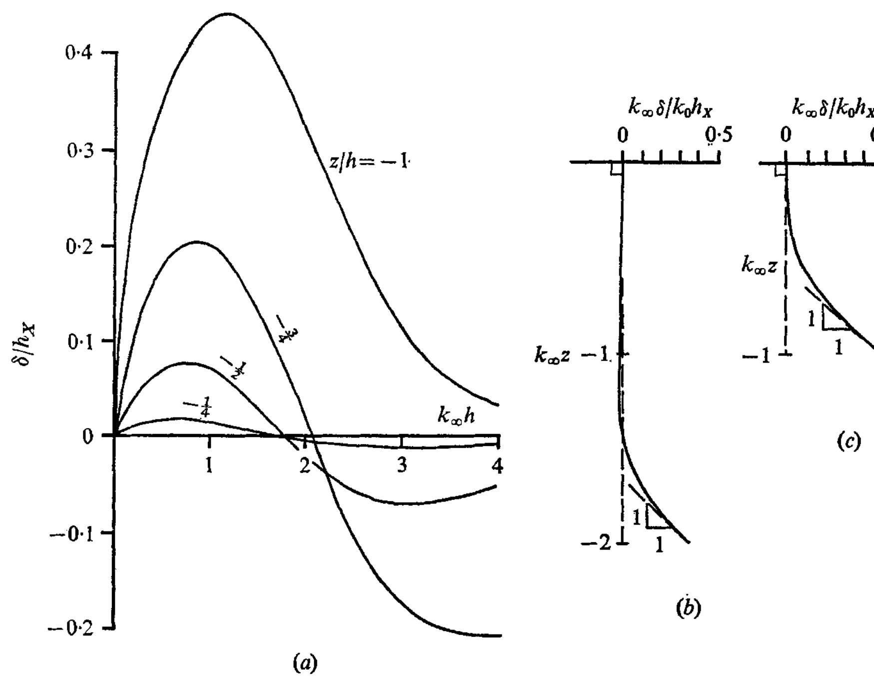

FIGURE 2. The phase change on a plane beach, normal incidence (a) δ/h_X vs. $k_\infty h$; $z/h = 0, -\frac{1}{4}, -\frac{1}{2}, -\frac{3}{4}, -1$. (b) $k_\infty \delta/k_0 h_X$ vs. $k_\infty z$; $k_\infty h = 2$. (c) $k_\infty \delta/k_0 h_X$ vs. $k_\infty z$; $k_\infty h = 1$.

We mention finally that for a completely submerged bottom of slowly varying depth, no perturbation theory for weak reflexion is yet known comparable to that of Bremmer (1949) for linear long waves (see also Kajiura 1962).

6. Linearized instability theory of Stokes waves

In the pioneering studies of Benjamin & Feir (1967), Benjamin (1967) and Whitham (1967), side-band disturbances were treated two-dimensionally. Benney & Roskes (1970), using the method of multiple scales, made several generalizations among which the modulation wave was taken to be oblique with respect to the Stokes wave train. New side-band cut-off limits were found. The same problem can be equally well treated on the basis of (4.3), (4.5), (4.6), (4.16) in the manner of Whitham (1967*a*). We take h = constant and introduce the notations $\mathbf{U} = \nabla\phi^{(1,0)}$ and $d = \eta^{(2,0)}$. Allowing small disturbances to a, $\mathbf{k}$, $\mathbf{U}$ and d so that

$$\left.\begin{aligned} a = \bar{a} + a', \quad \mathbf{k} = \bar{\mathbf{k}} + \mathbf{k}', \quad \mathbf{U} = \bar{\mathbf{U}} + \mathbf{U}', \quad d = \bar{d} + d' \\ \text{with} \qquad \mathbf{k}' = (\mu', \nu') \quad \text{and} \quad \mathbf{U}' = (U', V'). \end{aligned}\right\} \tag{6.1}$$

Consider the Stokes wave to be propagating in the x direction only, i.e.

$$\bar{\mathbf{k}} = (\bar{k}, 0) \quad \bar{\mathbf{U}} = (\bar{U}, 0). \tag{6.2}$$

Linearizing (4.3), (4.5), (4.6) and (4.16) with respect to the primed quantities, and dropping the symbol (–), for the constant state (Stokes),

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{2}{a}a' - \frac{C_g}{\omega_0}\mu'\right] + \frac{\partial}{\partial x}\left[\left(\frac{dC_g}{dk} - \frac{C_g^2}{\omega_0}\right)\mu' + \frac{2C_g}{a}a'\right] + \frac{\partial}{\partial y}\left[\frac{C_g}{k}\nu'\right] = 0,\\
&\frac{\partial d'}{\partial t} + \frac{\partial}{\partial x}\left[-\frac{g}{2}\frac{a^2kC_g}{\omega_0^2}\mu' + g\frac{ka}{\omega_0}a'\right] + \nabla.\left[h\mathbf{U}' + \frac{ga^2}{\omega_0}\mathbf{k}'\right] = 0,\\
&\frac{\partial \mathbf{U}}{\partial t} + \nabla\left[gd' + gkD_0aa' + \frac{g}{2}\frac{d(kD_0)}{dk}a^2\mu'\right] = 0,\\
&\frac{\partial \mathbf{k}'}{\partial t} + \nabla\left[C_g\mu' + \epsilon^2\omega_0k^2aS_0a' + \epsilon^2kU' + \epsilon^2\omega_0kD_0d'\right.\\
&\qquad\left. -\epsilon^2\left(\frac{dC_g}{dk} + \frac{C_g^2}{\omega_0}\right)\frac{a'_{xx}}{2a} - \epsilon^2\frac{C_g}{2ka}a'_{yy} + \epsilon^2\frac{a'_{tt}}{2\omega_0 a}\right] = 0,
\end{aligned}\right\} \tag{6.3a, b, c, d}$$

where
$$D_0 = \frac{1}{2}\left(\sigma - \frac{1}{\sigma}\right) \quad\text{and}\quad S_0 = \tfrac{1}{8}(9\sigma^4 - 10\sigma^2 + 9). \tag{6.4}$$

Assuming solutions of the type $\Delta\exp[i(\mathbf{K}.\mathbf{X} - \Omega t)]$, $\mathbf{K} = (K_1, K_2)$ an eigenvalue condition is obtained, yielding six eigenfrequencies. Four of them are real and hence stable to the order $O(\epsilon)$: $\Omega = \pm O(\epsilon^2)$ and $\Omega = \pm\sqrt{\{gh(K_1^2 + K_2^2)\}} + O(\epsilon)$, while the remaining two may be complex:

$$\begin{aligned}
\frac{\Omega}{K_1} &= C_g + \epsilon C_1\\
&= C_g \pm \epsilon\frac{1}{2}\frac{\omega_0}{k}\mathscr{Y}^{\frac{1}{2}}\left[\left(\frac{K_1}{k}\right)^2\mathscr{Y} + 2(ka)^2\mathscr{X}\right]^{\frac{1}{2}} + O(\epsilon^2),
\end{aligned} \tag{6.5}$$

where
$$\left.\begin{aligned}
\mathscr{X} &= S_0 + \frac{(g/\omega_0)^2 + 2(g/\omega_0)D_0C_g + ghD_0\sec^2\theta}{C_g^2 - gh\sec^2\theta},\\
\mathscr{Y} &= \frac{kC_g}{\omega_0}\left(\frac{k}{C_g}\frac{dC_g}{dk} + \tan^2\theta\right),\\
\theta &= \tan^{-1}\left(\frac{K_2}{K_1}\right).
\end{aligned}\right\} \tag{6.6a, b, c}$$

The growth rate is simply

$$\left(\frac{\partial}{\partial t} + C_g\frac{\partial}{\partial x}\right)\ln\Delta = -i\epsilon K_1C_1. \tag{6.7}$$

Detailed discussions on the region of instability in the $K_1 \sim K_2$ plane have been given by Benney & Roskes (1970) and will be omitted here. We point out that for $\theta = 0$ ($K_2 = 0$), (6.5)–(6.7) are reducible to Benjamin (1967, equation (46)) while Whitham's (1967a, equation (57)) result is obtained by further letting $K_1 = 0$.

7. Non-linear features of the modulation equations, permanent envelopes

The presence of the new terms ξ_X and ξ_T which give rise to triple derivatives and hence, represent dispersion, alters fundamentally the mathematical characteristics of the system of equations governing the slow modulation. In particular Whitham's original view on their hyperbolicity or ellipticity, and the possibility of shocks must now be revised. This change of mathematical property due to a weak dispersion term is a familiar one in shallow water theory as in Kortweg-deVries (KdV) or Boussinesq equations. Recent numerical work of Zabusky & Kruskal (1965) on the KdV equation has shown that the steepening of a wave profile due to nonlinearity is counteracted by the dispersion term so that shocks do not occur. This is not to say that shocks will not occur in reality; it merely indicates that before the formation of shocks, if any, terms of higher derivatives must become effective which changes any prediction on the position of shock by the non-dispersive theory.

To simplify further analysis, we restrict to the case of two dimensions and infinitely deep water ($kh \sim \infty$) which has been shown to be unstable according to a linearized analysis. The modulation equations reduce to two:

$$\left.\begin{aligned}&\frac{\partial}{\partial t}\left\{\frac{a^2}{\omega_0}\right\}+\frac{\partial}{\partial x}\left\{C_g\frac{a^2}{\omega_0}\right\}=0,\\&\frac{\partial k}{\partial t}+\frac{\partial}{\partial x}\left\{\omega_0\left[1+\epsilon^2\left(\frac{k^2a^2}{2}+\frac{(a/\omega_0)_{tt}}{2\omega_0 a}\right)\right]\right\}=0.\end{aligned}\right\}\qquad(7.1)$$

There is a structural resemblance of this set of equations to those of Boussinesq for long waves. To make it more evident we introduce $\mathscr{E}=\frac{1}{2}g(a^2/\omega_0)$ and rewrite (7.1 a,b) for two unknowns $\mathscr{E}$ and C_g.

$$\left.\begin{aligned}&\frac{\partial\mathscr{E}}{\partial t}+\frac{\partial}{\partial x}\{C_g\mathscr{E}\}=0,\\&\frac{\partial C_g}{\partial t}+C_g\frac{\partial C_g}{\partial x}-\epsilon^2\frac{\partial}{\partial x}\left\{\left(\frac{g}{32C_g^3}\right)\mathscr{E}+\left(\frac{2C_g^4}{g^2}\right)\frac{(\mathscr{E}C_g)^{\frac{1}{2}}_{tt}}{(\mathscr{E}C_g)^{\frac{1}{2}}}\right\}=0.\end{aligned}\right\}\qquad(7.2)$$

In Boussinesq equations the dispersion terms of $O(\epsilon^2)$ are linear, but otherwise all other terms are of the same form (see e.g. Whitham 1965a). Thus many features of Boussinesq equations may be anticipated here also.

We first look for modulational waves of permanent form $f(\xi)$, $\xi=x-\bar{C}t$. Equations (7.1 a,b) become

$$\left.\begin{aligned}&(C_g-\bar{C})\frac{a^2}{w_0}=\text{constant},\\&\left(1-\frac{\bar{C}}{C_g}\right)\omega_{0\xi}+\epsilon^2\left[\tfrac{1}{2}\omega_0(ka)^2+\frac{\bar{C}^2}{2a}\left(\frac{a}{\omega_0}\right)_{\xi\xi}\right]_\xi=0.\end{aligned}\right\}\qquad(7.3a,b)$$

Now in order that the convective term balance the dispersion terms in (7.3b), we should have $C_g-\bar{C}=O(\epsilon)$ and $\omega_{0\xi}=O(\epsilon)$. This suggests the following approximation:

$$C_g=\bar{C}+\epsilon C^{(1)}+\ldots,\quad \omega_0=\bar{\omega}+\epsilon\omega^{(1)}+\ldots,\qquad(7.4a,b,c)$$

where $$\bar{C} = g/2\bar{\omega}.$$

Since $C_g = g/2\omega$, it follows that

$$C_g - \bar{C} = \epsilon C^{(1)} = -\epsilon \frac{g}{2\bar{\omega}^2}\omega^{(1)}. \tag{7.5}$$

Using (7.4) and (7.5) and retaining leading terms only (7.3) may be simplified and integrated, which finally gives

$$[\bar{\mathscr{E}}_\xi]^2 = -\left(\frac{g^4}{32\bar{C}^4}\right)\bar{\mathscr{E}}^3 + C_1\bar{\mathscr{E}}^2 + C_2\bar{\mathscr{E}} + C_3, \tag{7.6}$$

where $$\bar{\mathscr{E}} = \frac{g}{2}\frac{a^2}{\bar{\omega}} = \frac{E}{\bar{\omega}} \tag{7.7}$$

is the wave 'action' and C_1, C_2, C_3 are constants. Equation (7.6) is known to give cnoidal and solitary waves as solutions. Similar conclusions have been reached by Benney & Newell (1967).

The existence of permanent envelopes supports the claim that (7.1*a*, *b*) are similar to Boussinesq equations in that both can support solutions for which non-linearity and dispersion are in balance.

Some facts about transient solutions of Boussinesq and Korteweg–deVries equations are pertinent here. Gardner *et al.* (1967) have discovered analytically that, according to the KdV equation, any initial disturbance vanishing at $|x| \to \infty$ will eventually disperse into a train of solitary waves of successively smaller amplitudes. The prevalence of solitary waves is also seen in numerical studies of Zabusky & Kruskal (1965). Recently, based on a variant of Boussinesq equations extended to uneven bottom, Madsen & Mei found numerically (i) that a solitary wave, distorted by a submerged beach, gradually disintegrates into a train of solitary waves on a shelf (Madsen & Mei 1969*a*), and (ii) that, for a sinusoidal disturbance at one end of a horizontal channel $\eta|_{x=0} = a \sin \omega t$, multiple crests evolve within any given primary wavelength $\sqrt{(gh)}2\pi/\omega$, they advance downstream at different speeds, causing a non-linear beat phenomenon. Larger crests eventually outrun the smaller ones, and emerge as cnoidal waves at the front (Madsen & Mei 1969*b*). Most of these features have been observed experimentally for shallow water waves (Street *et al.* 1968; Galvin 1968). Now considering the analogy between Boussinesq equations and the modulation equations (7.1*a*, *b*), we may speculate that a wave packet, i.e. a pulse-shaped envelope of a periodic wave train, will in general disperse into a series of successively smaller envelope pulses; this is indeed one of the features observed experimentally by Feir (1967). Secondly, we also believe that the instability of Stokes waves is really a phenomenon of finite amplitude modulation of the envelope, which may be either transient (multiple crests) or steady (permanent envelopes); this is again consistent with the experimental records of Feir (see Benjamin 1967, p. 963). Quantitative affirmation must await numerical studies of (7.1*a*, *b*) and further experiments in long tanks.

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Appendix A. Formal solutions to (2.12)

As mentioned before, the problem for $n = 1$, $m = 0$ is only partially determined by the pertinent boundary-value problem. Here we have

$$R^{(1,0)} = F^{(1,0)} = G^{(1,0)} = H^{(1,0)} = 0; \tag{A 1}$$

hence,

$$\phi_z^{(1,0)} = 0, \quad \eta^{(1,0)} = 0. \tag{A 2a.b}$$

We recall that $n = 1$, $m = 1$ the homogeneous solution is

$$\phi^{(1,1)} = \frac{ig}{\omega_0} A^{(1,1)} \cosh Q \quad \eta^{(1,1)} = A^{(1,1)} \cosh q = \tfrac{1}{2}a \tag{A 3}$$

with k_0 and ω_0 satisfying (3.9).

For $n = 2, m = 1$, we have

$$\left.\begin{aligned} R^{(2,1)} &= -2i\mathbf{k}_0 . \nabla\phi^{(1,1)} - i(\nabla . \mathbf{k}_0)\,\phi^{(1,1)}, \\ F^{(2,1)} &= -\nabla h . \{i\mathbf{k}_0 \phi^{(1,1)}\}_{z=-h}, \\ G^{(2,1)} &= \{i(\omega_0\phi^{(1,1)})_t + i\omega_0\phi_t^{(1,1)}\}_{z=0}, \\ H^{(2,1)} &= \{\phi_t^{(1,1)}\}_{z=0}. \end{aligned}\right\} \tag{A 4a, b, c, d}$$

The solution for $\phi^{(2,1)}$ then follows from (3.6):

$$\phi^{(2,1)} = A^{(2,1)} \cosh Q - (gA^{(1,1)}/\omega_0)\,\{\alpha_1 Q \cosh Q + \alpha_2 Q \sinh Q + \alpha_3 Q^2 \cosh Q\}, \tag{A 5}$$

where

$$\alpha_1 = \frac{\mathbf{k}_0}{k_0}.\nabla h, \quad \alpha_2 = \frac{\nabla . [(\mathbf{k}_0/k_0)\,(A^{(1,1)}/\omega_0)^2]}{2k_0(A^{(1,1)}/\omega_0)^2}, \quad \alpha_3 = \frac{\mathbf{k}_0}{2k_0^3}.\nabla k_0. \tag{A 6a, b, c}$$

The coefficient $A^{(2,1)}$ for the homogeneous solution will be chosen here so as to give the proper limit as $k_0 h \to \infty$ (which can be worked out independently); note that this requirement still does not give a unique choice for $A^{(2,1)}$. We take

$$\phi^{(2,1)} = -\frac{gA^{(1,1)}}{\omega_0}\,[\alpha_1(Q-q) + \alpha_2(Q \tanh Q - q \tanh q) + \alpha_3(Q^2 - q^2)] \cosh Q \tag{A 7}$$

so that $[\phi^{(2,1)}]_{z=0} = 0$. This leads to

$$\eta^{(2,1)} = -\frac{1}{g}\,[\phi_t^{(1,1)}]_{z=0} = \frac{i}{2}\left(\frac{a}{\omega_0}\right)_t. \tag{A 8}$$

For $n = 2, m = 0$, we have

$$\begin{aligned} R^{(2,0)} &= F^{(2,0)} = G^{(2,0)} = 0, \\ H^{(2,0)} &= \tfrac{1}{4}gk_\infty(\sigma^2 - 1)\,a^2 + \phi_t^{(1,0)} \end{aligned} \tag{A 9a, b}$$

where $\sigma = k_0/k_\infty$. Substituting into (2.12*d*),

$$g\eta^{(2,0)} = -\tfrac{1}{4}gk_\infty a^2\,(\sigma^2 - 1) - \phi_t^{(1,0)}. \tag{A 10}$$

For $n = 2, m = 2$,

$$\left.\begin{aligned} R^{(2,2)} &= F^{(2,2)} = 0, \\ G^{(2,2)} &= \tfrac{3}{4}i\omega_0\, gk_\infty a^2\,(\sigma^2 - 1), \\ H^{(2,2)} &= \tfrac{1}{8}gk_\infty a^2\,(\sigma^2 - 3) \end{aligned}\right\} \tag{A 11a, b, c}$$

which lead to the solution

$$\left.\begin{aligned} \phi^{(2,2)} &= -\tfrac{3}{16}i\omega_0 a^2(\sigma^2 - 1)^2 \cosh 2Q, \\ \eta^{(2,2)} &= \tfrac{1}{8}(k_\infty a^2)\sigma^2(3\sigma^2 - 1). \end{aligned}\right\} \tag{A 12a, b}$$

Appendix B

The explicit formula for ξ_X is given for general $h(x)$, and $\omega = \omega_0 =$ constant. We remark that it is this term which requires much more tedious algebra in comparison with Whitham's analysis where this contribution is neglected:

$$\begin{aligned}\xi_X = & -\frac{\alpha_1\alpha_2}{\sinh 2q} - \frac{1}{4}\left(\beta_1 - \frac{2\delta_x}{k_0}\right)\left(1+\frac{2q}{\sinh 2q}\right) \\ & -\tfrac{1}{4}\beta_2\left(\coth 2q - \frac{1}{\sinh 2q}\right) - \tfrac{1}{8}\beta_3\left(2q - \coth 2q + \frac{1+2q^2}{\sinh 2q}\right) \\ & -\tfrac{1}{8}\beta_4(2q\coth 2q - 1) + \tfrac{1}{8}\beta_5\left(2q^2 - 2q\coth 2q + 1 + \frac{4q^3}{3\sinh 2q}\right) \\ & -\tfrac{1}{8}\beta_6\left[(1+2q^2)\coth 2q - 2q - \frac{1}{\sinh 2q}\right] \\ & -\tfrac{1}{16}\beta_7[(4q^3+6q)\coth 2q - (6q^2+3)],\end{aligned} \tag{B 1}$$

where

$$\left.\begin{aligned}\beta_1 &= 3\alpha_1^2 + \alpha_6, \\ \beta_2 &= 4\alpha_1\alpha_2 + 4\alpha_1\alpha_3 + \alpha_4, \\ \beta_3 &= 14\alpha_1\alpha_3 + 4\alpha_1\alpha_2 + 2\alpha_4, \\ \beta_4 &= 6\alpha_2\alpha_3 + 2\alpha_1^2 + 2\alpha_5 + 2\alpha_6, \\ \beta_5 &= 2\alpha_5 + 6\alpha_2\alpha_3 + 6\alpha_3^2, \\ \beta_6 &= 6\alpha_1\alpha_3, \\ \beta_7 &= 4\alpha_3^2,\end{aligned}\right\} \tag{B 2}$$

with (cf. A 6)

$$\left.\begin{aligned}&\alpha_1 = h_x, \quad \alpha_2 = \frac{(A^{(1,1)}/\omega_0)_x}{k_0(A^{(1,1)}/\omega_0)}, \quad \alpha_3 = \frac{k_{0x}}{2k_0^2}, \\ &\alpha_4 = \frac{h_{xx}}{k_0}, \quad \alpha_5 = \frac{k_{0xx}}{2k_0^3}, \quad \alpha_6 = \frac{(A^{(1,1)}/\omega_0)_{xx}}{k_0^2(A^{(1,1)}/\omega_0)}.\end{aligned}\right\} \tag{B 3}$$

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